

Lecture 21

Hierarchy Theorems

Machine as Strings and Universal TM

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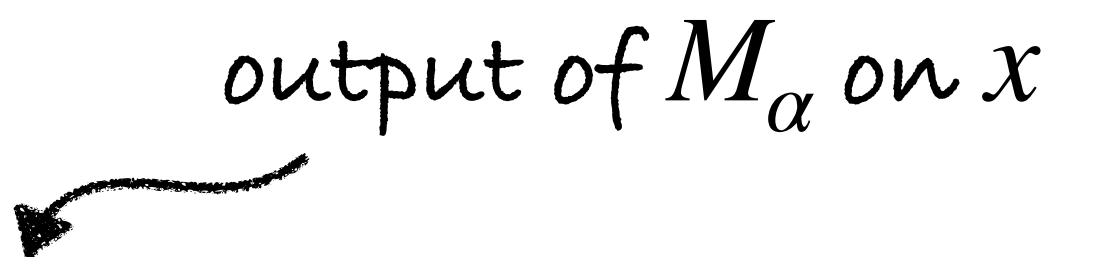
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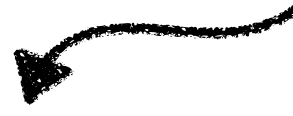
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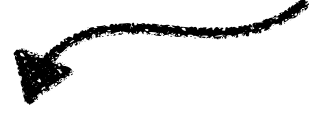
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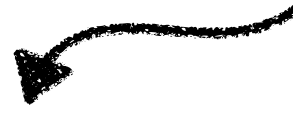
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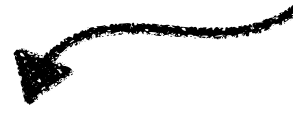
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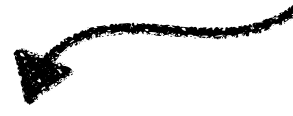
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Diagonalization is used in **Hierarchy theorems** and **Ladner's theorem**.

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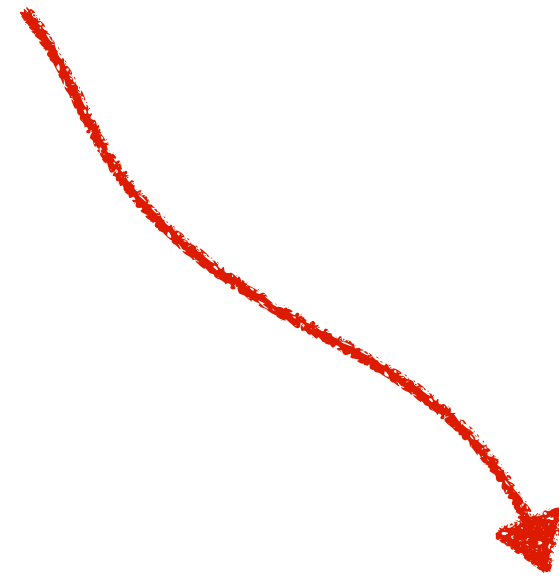
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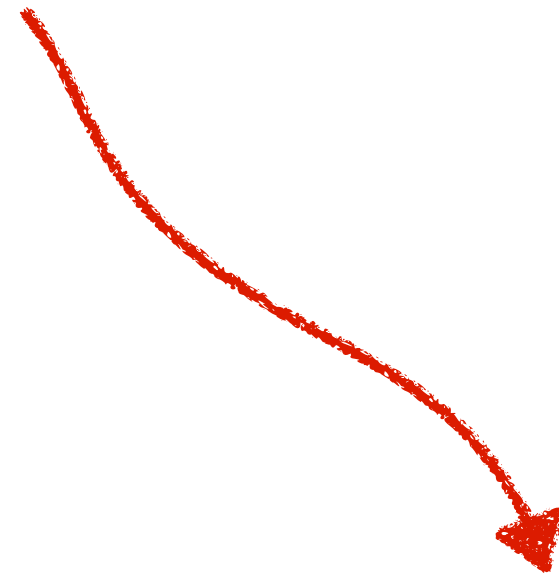
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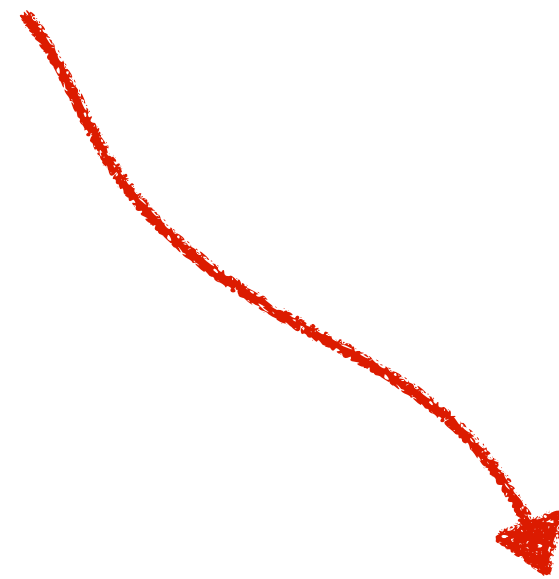
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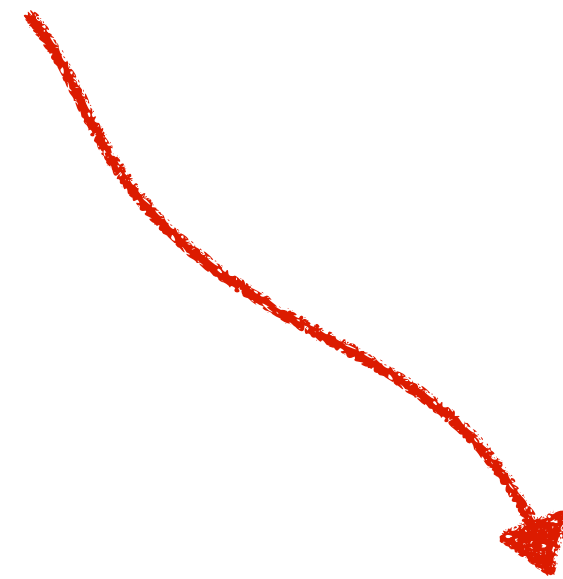
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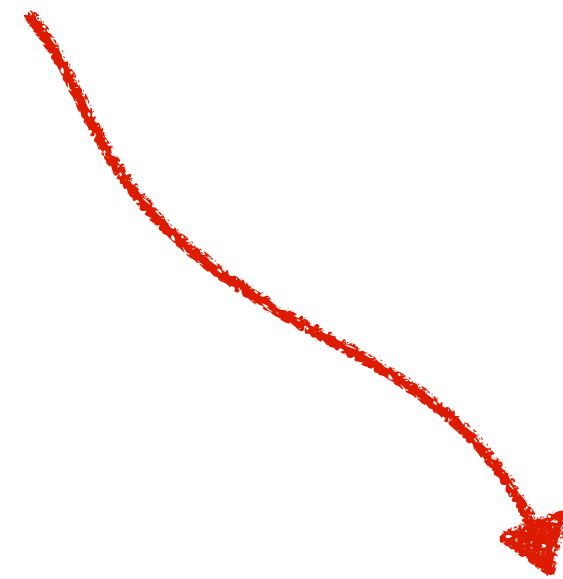
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- Runs UTM U on (x, x) for $g(|x|)$ steps.
- If M_x halts on x and writes some bits on the output tape within this time, then D outputs the opposite of the first bit.
- Else, D outputs 0.

Deterministic Time Hierarchy

Claim 2: $L(D) \notin \text{DTIME}(f(n))$.

Proof: Suppose \exists a DTM M with running time $O(f(n))$ that decides $L(D)$.

- M on any input x halts within $cf(|x|)$ steps, where c is a constant.
- UTM U can simulate M on input x within $c'f(|x|)\log f(|x|)$ steps, where c' is a constant...

We know that $\exists n_0$ such that $\forall n \geq n_0, c'f(n)\log f(n) < g(n)$. ($\because f(n)\log f(n) = o(g(n))$)

Let x be a binary representation of M whose length is at least n_0 .

What happens when D gets x as input?

- M_x halts on x within $g(|x|)$ steps of U .
- If M_x accepts x , then D rejects x .
- If M_x rejects x , then D accepts x .

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